

# KANTOWSKI-SACHS BRANE COSMOLOGY

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We consider brane Kantowski-Sachs Universe when bulk space is five-dimensional Anti-deSitter space. The corresponding cosmological equations with perfect fluid are written. For several specific choices of relation between energy and pressure it is found the behavior of scale factors at early time. In particular, for  $\gamma = 3/2$  Kantowski-Sachs brane cosmology is modified to become the isotropic one, while for  $\gamma = 1$  it remains the anisotropic cosmology in the process of evolution.

## 1. Introduction

Since the work by Randall-Sundrum [1] it was realized that brane cosmology is quite similar to the standard four-dimensional cosmology at late times. However, it may significantly differ from standard cosmology at early times. Nevertheless, the brane gravity shows the newtonian behaviour despite the fact that braneworld is five-dimensional one.

In the standard as well as in brane cosmology it is quite possible that the early Universe could be the anisotropic one. During the evolution the anisotropy stage should quickly be changed by the isotropic stage due to the classical or quantum matter effects, or due to the modification of the gravity theory or by some other phenomena. In the present paper we discuss the brane Kantowski-Sachs (KS) cosmology and compare it with the standard Kantowski-Sachs cosmology in Einstein gravity for some specific matter choice. It is shown that brane KS cosmology may quickly become isotropic one for some choices of matter.

## 2. Standard versus brane Kantowski-Sachs cosmology

2.0. We start from the five-dimensional braneworld which is defined by the condition  $Y(X^I) = 0$ , where  $I = 0, 1, 2, 3, 4$  are 5-dimensional coordinates. The starting action in five dimensional space is [2, 3]

$$S = \int d^5X \sqrt{-g_5} \left( \frac{1}{2k_5^2} R_5 - \Lambda_5 \right) + \int_{Y=0} d^4x \sqrt{-g} \left( \frac{1}{k_5^2} K^\pm - \lambda + L^{matter} \right). \quad (1)$$

with  $k_5^2 = 8\pi G_5$  being the 5-dimensional gravitational coupling constant and  $x^\mu$ , ( $\mu = 0, 1, 2, 3$ ) are the induced 4-dimensional brane coordinates.  $R_5$  is the 5D intrinsic curvature in the bulk and  $K^\pm$  is the intrinsic curvature on either side of the brane.

The 5D Einstein equation has form

$$^{(5)}G_{IJ} = k_5^2 \ ^{(5)}T_{IJ},$$

$$^{(5)}T_{IJ} = -\Lambda_5 \ ^{(5)}g_{IJ} + \delta(Y) [-\lambda g_{IJ} + T_{IJ}^{matter}].$$

Assuming a metric of the form  $ds^2 = (n_I n_J + g_{IJ}) dx^I dx^J$ , with  $n_I dx^I = d\xi$  the unit normal to the  $\xi = \text{const}$  hypersurfaces and  $g_{IJ}$  the induced metric on  $\xi = \text{const}$  hypersurfaces, the effective four-dimensional gravitational equations on the brane take the form [2, 3, 4]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu}, \quad (2)$$

where

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_\mu^\alpha T_{\alpha\nu} + \frac{1}{24} g_{\mu\nu} (3T^{\alpha\beta} T_{\alpha\beta} - T^2),$$

and  $\Lambda = k_5^2(\Lambda_5 + k_5^2 \lambda^2/6)$ ,  $k_4^2 = k_5^2 \lambda^2/6$  and  $E_{IJ} = C_{IAJB} n^A n^B$ .  $C_{IAJB}$  is the 5-dimensional Weyl tensor in the bulk and  $\lambda$  is the vacuum energy on the brane.  $T_{\mu\nu}$  is the matter energy-momentum tensor on the brane with components  $T_0^0 = -\rho$ ,  $T_1^1 = T_2^2 = T_3^3 = p$  and  $T = T_\mu^\mu$  is the trace of the energy-momentum tensor. One chooses  $p = (\gamma - 1)\rho$ , hence  $1 \leq \gamma \leq 2$ .

From the equation (2) it follows that there exist several cases:

- 1) Conventional Einstein Theory (CET) which is 4D and
- 2) Brane Cosmology (BC) which includes 5D effects.

These cases originate from different relations between CET and brane world scenario which can lead to two type of corrections: (a) the matter fields contribute local "quadratic" energy-momentum correction via the tensor  $S_{\mu\nu}$ , and (b) the "nonlocal" effects via bulk Weyl tensor.

We will consider the brane metric in the Kantowski-Sachs form [5, 6].

$$ds^2 = -dt^2 + a_1(t)^2 dr^2 + a_2(t)^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

The following variables are convenient to introduce:

$$\begin{aligned} V &= a_1 a_2^2, \\ H_i &= \frac{\dot{a}_i}{a_i}, \\ H &= \frac{1}{3}(H_1 + 2H_2) = \frac{\dot{V}}{3V}. \end{aligned}$$

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2.1. In this case the Einstein equation looks:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu}, \quad \nabla_\mu T^{\mu\nu} = 0, \quad (3)$$

with  $G_{\mu\nu}$  the Einstein tensor (4D),  $\Lambda$  the cosmological constant,  $k_4$  gravitational coupling  $k_4^2 = 8\pi G$ . These equations for Kantowski-Sachs Universe become:

$$\begin{aligned} \frac{\dot{a}_2^2}{a_2^2} + 2\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2} &= \Lambda + k_4^2\rho, \\ 2\frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} &= \Lambda + k_4^2\rho(1-\gamma), \\ \frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} &= \Lambda + k_4^2\rho(1-\gamma), \\ \dot{\rho} + \gamma\rho\left(\frac{\dot{a}_1}{a_1} + 2\frac{\dot{a}_2}{a_2}\right) &= 0. \end{aligned} \quad (4)$$

The Eq. (4) can be easily solved to describe the time evolution law of the energy density of the fluid:

$$\rho = \rho_0 V^{-\gamma}, \quad \rho_0 = \text{constant} > 0.$$

One can rewrite this equation in another form:

$$\frac{d}{dt}(VH_1) = \Lambda V + \frac{1}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma), \quad (5)$$

$$\frac{d}{dt}(VH_2) = \Lambda V + \frac{1}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) - a_1, \quad (6)$$

$$3\dot{H} + H_1^2 + 2H_2^2 = \Lambda + \frac{1}{2}k_4^2\rho_0 V^{-\gamma}(2-3\gamma). \quad (7)$$

From the first two equations the equation for  $V$  maybe written:

$$\ddot{V} = 3\Lambda V + \frac{3}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) - 2a_1. \quad (8)$$

The equation can be partly integrated:

$$\dot{V} = \sqrt{3\Lambda V^2 + 3k_4^2\rho_0 V^{2-\gamma} - 2\int a_1 dV}. \quad (9)$$

From (5) and (6) it follows that:

$$H_1 = H + \frac{2}{3V}K,$$

$$H_2 = H - \frac{1}{3V}K,$$

$$K = \int a_1 dt.$$

If we substitute these equations into (7) then:

$$-2a_1V + \frac{4}{3}\int a_1 dV + \frac{2}{3}\left(\int a_1 dt\right)^2 = 0. \quad (10)$$

Let us consider the asymptotic behaviour. If  $V$  is large then solution has the simple form

$$\begin{aligned} V &= a e^{\sqrt{3\Lambda}t}, \\ a_1 &= \frac{a}{b^2} e^{\sqrt{\Lambda/3}t}, \quad a_2 = b e^{\sqrt{\Lambda/3}t}, \\ H_1 &= H_2, \end{aligned} \quad (11)$$

here  $a$  and  $b$  are constants,  $\Lambda$  is not zero. The behaviour of the system does not depend on  $\gamma$  and the anisotropic CET Universe becomes isotropic one for large  $V$  due to classical matter effects.

The situation for extremely small  $V$  is different. The properties of the CET Universe will depend on the value of  $\gamma$ . From Eq. (9)

$$\dot{V} = \sqrt{3k_4^2\rho_0 V^{2-\gamma} + b}. \quad (12)$$

For  $\gamma = 3/2$  the solution takes the form

$$\begin{aligned} a_1 &= \frac{9a^{2/3}(t-t_0)^{4/9}}{4c^2}, \\ a_2 &= c(t-t_0)^{4/9}, \\ V &= \frac{9}{4}a^{2/3}(t-t_0)^{4/3}. \end{aligned} \quad (13)$$

Here  $c$  is the constant of integration and  $a = \frac{1}{2}k_4^2\rho_0$ . For  $\gamma = 1$  the solution can not be found in analytical form. Hence, we finished the review of the anisotropic CET Universe. It is found that even for small cosmological time the process of isotropisation quickly starts.

2.2. In this subsection we consider the brane cosmology with zero Weyl tensor which is natural for 5D AdS bulk. The Einstein equations and evolution law of the energy density take the form

$$\begin{aligned} \frac{\dot{a}_2^2}{a_2^2} + 2\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2} &= \\ &= \Lambda + k_4^2\rho + \frac{1}{12}k_5^4\rho^2, \\ 2\frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} &= \\ &= \Lambda + k_4^2\rho(1-\gamma) + \frac{1}{12}k_5^4\rho^2(1-2\gamma), \\ \frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} &= \\ &= \Lambda + k_4^2\rho(1-\gamma) + \frac{1}{12}k_5^4\rho^2(1-2\gamma), \\ \dot{\rho} + \gamma\rho\left(\frac{\dot{a}_1}{a_1} + 2\frac{\dot{a}_2}{a_2}\right) &= 0. \end{aligned} \quad (14)$$

The Eq. (14) can be easily solved:

$$\rho = \rho_0 V^{-\gamma}, \quad \rho_0 = \text{constant} > 0.$$

We can rewrite above equations in another form:

$$\begin{aligned} \frac{d}{dt}(VH_1) &= \Lambda V + \frac{1}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) + \\ &+ \frac{1}{12}k_5^4\rho_0^2 V^{1-2\gamma}(1-\gamma), \\ \frac{d}{dt}(VH_2) &= \Lambda V + \frac{1}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) + \\ &+ \frac{1}{12}k_5^4\rho_0^2 V^{1-2\gamma}(1-\gamma) - a_1, \end{aligned}$$

$$\begin{aligned}
3\dot{H} + H_1^2 + 2H_2^2 &= \Lambda + \frac{1}{2}k_4^2\rho_0 V^{-\gamma}(2-3\gamma) + \\
&+ \frac{1}{12}k_5^4\rho_0^2 V^{-2\gamma}(1-3\gamma).
\end{aligned}$$

From the first two equations the equation for  $V$  can be obtained:

$$\begin{aligned}
\ddot{V} &= 3\Lambda V + \frac{3}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) + \\
&+ \frac{1}{4}k_5^4\rho_0^2 V^{-2\gamma}(1-\gamma) - 2a_1.
\end{aligned} \tag{15}$$

It is easy to show that from these equations we can get formally the same equations that in Conventional Einstein's Theory:

$$\begin{aligned}
H_1 &= H + \frac{2}{3V}K, \\
H_2 &= H - \frac{1}{3V}K, \\
K &= \int a_1 dt, \\
-2a_1 V + \frac{4}{3} \int a_1 dV + \frac{2}{3} \left( \int a_1 dt \right)^2 &= 0.
\end{aligned}$$

The asymptotic behaviour for large  $V$  has the same form as in CET. However, for extremely small  $V$  one obtains the difference in comparison with CET.

For  $\gamma = 1$

$$\begin{aligned}
a_1 &= \frac{1}{c_2^2}(c_1 + \sqrt{a^2 + bt})^{1/3 - \frac{2\sqrt{b}}{3\sqrt{a^2 + b}}}, \\
a_2 &= c_2(c_1 + \sqrt{a^2 + bt})^{1/3 + \frac{\sqrt{b}}{3\sqrt{a^2 + b}}}, \\
V &= c_1 + \sqrt{a^2 + bt}.
\end{aligned}$$

Thus, unlike to the situation with CET the analytical solution appears. Nevertheless, the solution remains anisotropic KS cosmology.

For  $\gamma = 3/2$

$$\begin{aligned}
a_1 &= \frac{1}{c_2^2} \left( \frac{3a}{2} \right)^{2/3} (t - t_0)^{2/9}, \\
a_2 &= c_2(t - t_0)^{2/9}, \\
V &= \left( \frac{3a}{2} \right)^{2/3} (t - t_0)^{2/3}.
\end{aligned}$$

Here  $b, t_0, c_2$  are constants,  $a^2 = \frac{1}{4}k_5^4\rho_0^2$ .

### 3. Discussion

In summary, we compared the KS brane cosmology with the KS cosmology for Einstein theory for several classical matter choices.

We demonstrated that with  $\gamma = 1$  (where CET anisotropic Universe does not have the analytical asymptotic), brane KS Universe remains anisotropic. For

$\gamma = 3/2$ , brane KS Universe becomes isotropic at early times as well as analogous CET case. However, the details of isotropisation (scale factors) are slightly different, what shows the role of five-dimensional bulk.

One can also consider the simple case, when  $a_1 = \text{const.}$  For CET we get the exact solution

$$a_2 = \pm \sqrt{\frac{k_4^2\rho_0}{a_1^2} - t^2 + 2tc - c^2}$$

Here  $c$  is an integration constant,  $\Lambda = 0$ ,  $\gamma = 2$ . However, for brane cosmology there is no any exact solution. This shows the qualitative difference between CET and brane cosmology.

It would be really interesting to understand the role of quantum effects like in Brane New World scenario [7] to above KS brane cosmology. In particular, it is expected that quantum effects may lead to isotropic cosmology even for initially anisotropic brane Universe as it happens for standard Einstein theory [8]. Another interesting topic could be the generalisation of about discussion for brane wormholes (see [9] for recent discussion).

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